

IDEALS GENERATED BY 2-MINORS OF HANKEL MATRICES

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ABSTRACT. We study ideals generated by 2-minors of generic Hankel matrices.

INTRODUCTION

In [1], the authors introduced and studied binomial edge ideals associated with scrolls. More precisely, to a closed graph G on the vertex set $[n]$ with the edge set $E(G)$, one associates the binomial ideal $I_G \subset K[x_1, \dots, x_{n+1}]$ generated by the 2-minors of the $(2 \times n)$ - Hankel matrix $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \end{pmatrix}$ which correspond to the edges of the graph G . In other words, $I_G = \left(\begin{vmatrix} x_i & x_j \\ x_{i+1} & x_{j+1} \end{vmatrix} : i < j, \{i, j\} \in E(G) \right)$.

The definition of scroll binomial edge ideals was inspired by the construction of the classical binomial edge ideals as they were introduced by [6] and [7] a few years ago. Later on, there were considered several ways to generalize classical binomial edge ideals. We refer the reader to [4, 5, 8] for further information on these generalizations. Similar developments may be considered for scroll binomial edge ideals. One direction of generalization is illustrated in this paper.

Namely, for a generic Hankel matrix $X = (x_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$, with $x_{ij} = x_{i+j-1}$ for all i, j , and for two closed graphs G_1 on the vertex set $[m]$ with edge set $E(G_1)$, and G_2 on the vertex set $[n]$ with edge set $E(G_2)$, we consider the Hankel binomial ideal $I_{G_1, G_2} \subset S = K[x_1, \dots, x_{m+n-1}]$ defined as follows:

$$I_{G_1, G_2} = (g_{e,f} = \begin{vmatrix} x_{i+k-1} & x_{i+l-1} \\ x_{j+k-1} & x_{j+l-1} \end{vmatrix} : e = \{i, j\} \in E(G_1), f = \{k, l\} \in E(G_2)).$$

If G_1 and G_2 are complete graphs, that is, $G_1 = K_m$ and $G_2 = K_n$, then I_{G_1, G_2} is generated by all the 2-minors of X . We refer the reader to [2, 9] for information about the ideal I_{K_m, K_n} .

In this paper, we work with closed graphs. We recall from [6] that a simple graph G on the vertex set $[n]$ is closed if there exists a labeling of its vertices with the property that if $1 \leq i < j < k \leq n$ or $1 \leq k < j < i \leq n$, and $\{i, j\}, \{i, k\}$ are edges of G , then $\{j, k\}$ is an edge of G . This is equivalent to saying that if $\{i, j\} \in E(G)$ with $i < j$, then, for all $i < k < j$, $\{i, k\}$ and $\{k, j\}$ are edges in G .

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In [3], it was shown that a simple graph G on $[n]$ is closed if and only if there exists a labeling of G such that all facets of the clique complex $\Delta(G)$ of G are intervals $[a, b] \subset [n]$. A clique of G is a complete subgraph of G . The cliques of G form a simplicial complex called the clique complex of G .

Throughout this paper, if G is a closed graph on $[n]$, we assume that G is labeled such that if its maximal cliques are F_1, \dots, F_r , then $F_i = [a_i, b_i]$ for $1 \leq i \leq r$, and $1 = a_1 < a_2 < \dots < a_r < b_r = n$; see [3, Theorem 2.2]. In addition, we write $\Delta(G) = \langle F_1, \dots, F_r \rangle$ if the maximal cliques of G are F_1, \dots, F_r .

Let G_1, G_2 be connected closed graphs on $[m]$, respectively $[n]$. To G_1 and G_2 we associate a graph G on the vertex set $[m + n - 2]$ with the edge set:

$$E(G) = \{\{i + k - 1, j + l - 2\} : i < j, k < l, \{i, j\} \in E(G_1), \text{ and } \{k, l\} \in E(G_2)\}.$$

In Theorem 1.1, we show that $I_{G_1, G_2} = I_G$, where $I_G \subset S$ is the scroll binomial edge ideal of G . Moreover, G is a connected closed graph. This is the main result of our paper. It allows us to apply all the known results on scroll binomial edge ideals proved in [1]. The first consequence of Theorem 1.1 is that I_{G_1, G_2} has a quadratic Gröbner basis with respect to the revlexicographic order on S induced by $x_1 > \dots > x_{m+n-1}$. Additionally, it follows that I_{G_1, G_2} is a Cohen-Macaulay ideal of dimension 2.

In Proposition 2.1, we show that any maximal clique of the graph G is actually obtained by "adding" a maximal clique $[a, b]$ of G_1 with a maximal clique of G_2 . By using this proposition, in Theorem 2.3 we derive the main properties of I_{G_1, G_2} : primality, minimal primes, radical property, linear resolution.

Finally, in Proposition 2.4, we show that $\text{reg}(S/I_{G_1, G_2}) \leq m + n - 2$ and the equality holds if and only if G_1 and G_2 are line graphs.

We would like to make a final remark. If one of the graphs G_1, G_2 is not connected and the other one is connected, then the associated graph G is still connected, thus all the proved results are still valid. If both graphs are disconnected, then one easily sees that G might be disconnected. In that case, we may apply only the results of [1] which do not involve the connectedness of the graph G . We chose to treat only the case when G_1 and G_2 are connected since this is the most interesting setting and to avoid long technical arguments needed for distinguish between those graphs G_1 and G_2 which give a connected or disconnected graph G .

1. GRÖBNER BASIS

Let G_1, G_2 be two connected closed graphs on the vertex $[m]$ and $[n]$, respectively, and X be a generic $(m \times n)$ - Hankel matrix with $m \leq n$. Thus,

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_2 & x_3 & \dots & x_{n+1} \\ \dots & \dots & \dots & \dots \\ x_m & x_{m+1} & \dots & x_{m+n-1} \end{pmatrix}.$$

Let $e = \{i, j\} \in E(G_1)$ with $i < j$ and $f = \{k, l\} \in E(G_2)$ with $k < l$. To the pair (e, f) , we assign the following 2-minor of X :

$$g_{e,f} = [i \ j | k \ l] = x_{i+k-1} x_{j+l-1} - x_{j+k-1} x_{i+l-1}.$$

We fix a field K and let $S = K[x_1, \dots, x_{m+n-1}]$ endowed with the reverse lexicographic order induced by $x_1 > x_2 > \dots > x_{m+n-1}$. Then, with respect to this order, $\text{in}_{\text{rev}}(g_{e,f}) = x_{j+k-1}x_{i+l-1}$.

Let $X' = \begin{pmatrix} x_1 & x_2 & \dots & x_{m+n-2} \\ x_2 & x_3 & \dots & x_{m+n-1} \end{pmatrix}$ be the $2 \times (m+n-2)$ -Hankel matrix and G be the graph on the vertex set $[m+n-2]$ whose edge set is:

$$E(G) = \{\{i+k-1, j+l-2\} : i < j, k < l, \{i, j\} \in E(G_1) \text{ and } \{k, l\} \in E(G_2)\}.$$

Let G_1 and G_2 be as before. We define the Hankel ideal of the matrix X as

$$I_{G_1, G_2} = (g_{e,f} : e \in E(G_1), f \in E(G_2)).$$

In addition, let $I_G = (g_{ij} = \begin{vmatrix} x_i & x_j \\ x_{i+1} & x_{j+1} \end{vmatrix} : i < j, \{i, j\} \in E(G))$ be the scroll binomial edge ideal defined on the matrix X' .

With the above notation and settings we may state the main result of this paper.

Theorem 1.1. *Let G_1 and G_2 be closed graphs. Then G is a connected closed graph and $I_{G_1, G_2} = I_G$.*

Proof. Let $\{p, q\}, \{p, r\} \in E(G)$ and $q < r$. Then $p = i+k-1$, $q = j+l-2$ and $r = u+v-2$ for some $\{i, j\}, \{i, u\} \in E(G_1)$ and $\{k, l\}, \{k, v\} \in E(G_2)$. We may assume that $j < u$ and $l < v$. Then the $\{j-1, u\} \in E(G_1)$ and $\{l, v\} \in E(G_2)$ because G_1 and G_2 are closed. This gives $\{j+l-2, v+u-2\} = \{q, r\} \in E(G)$.

Similarly, if $\{p, q\}, \{r, q\} \in E(G)$ with $p < r < q$, then by similar arguments, it follows that $\{p, r\} \in E(G)$. Therefore, G is a closed graph. For connectedness, it is enough to observe that, for any $i \leq m-1$ and $k \leq n-1$, $\{i, i+1\} \in E(G_1)$ and $\{k, k+1\} \in E(G_2)$, thus $\{i+k-1, i+k\} \in E(G)$.

Next, we prove the equality $I_{G_1, G_2} = I_G$. Let $e = \{i, j\} \in E(G_1)$ and $f = \{k, l\} \in E(G_2)$ and $e_f = \{i+k-1, j+l-2\} \in E(G)$. Then $h_{e_f} = x_{i+k-1}x_{j+l-1} - x_{i+k}x_{j+l-2}$ and $g_{e,f} = x_{i+k-1}x_{j+l-1} - x_{i+l-1}x_{j+k-1}$ are typical generators of I_G and I_{G_1, G_2} , respectively. First, we show that $I_G \subset I_{G_1, G_2}$. If $j = i+1$ or $l = k+1$ we get $h_{e_f} = g_{e,f}$, thus $h_{e_f} \in I_{G_1, G_2}$. Now we consider $j > i+1$ and $l > k+1$. By using the fact that G_1 and G_2 are closed graphs, we see that $\{i, p\}, \{p, j\} \in E(G_1)$, and $\{k, q\}, \{q, l\} \in E(G_2)$ for all $i < p < j$ and $k < q < l$. In particular, $e' = \{i+1, j\} \in E(G_1)$ and $f' = \{k, l-1\} \in E(G_2)$. Then $g_{e', f'} = x_{i+k}x_{j+l-2} - x_{i+l-1}x_{j+k-1} \in I_{G_1, G_2}$ and $h_{e_f} = g_{e,f} - g_{e', f'} \in I_{G_1, G_2}$. Therefore, $I_G \subset I_{G_1, G_2}$.

Now, we show that $I_{G_1, G_2} \subset I_G$. Let $l-k > j-i = t$. Again, by using the fact the G_1 and G_2 are closed, we see that $e_1 = \{i+1, j\}, e_2 = \{i+2, j\}, \dots, e_t = \{i+t-1, j\} \in E(G_1)$ and $f_1 = \{k, l-1\}, f_2 = \{k, l-2\}, \dots, f_t = \{k, l-t+1\} \in E(G_2)$. Then $g_{e,f} = h_{e_f} + h_{e_1 f_1} + h_{e_2 f_2} + \dots + h_{e_t f_t}$, which gives $g_{e,f} \in I_G$. Similarly, one can show $g_{e,f} \in I_G$ when $j-i > l-k$. This completes the proof. \square

By applying [1, Theorem 1.1] and [1, Corollary 1.3] we get the following consequence of the above theorem.

Corollary 1.2. *Let G_1, G_2 be two connected closed graphs on the vertex sets $[m]$, respectively $[n]$. Then I_{G_1, G_2} has a quadratic Gröbner basis with respect to the revlexicographic order induced by $x_1 > \cdots > x_{m+n-1}$. Moreover, I_{G_1, G_2} is a Cohen-Macaulay ideal of dimension 2.*

2. PROPERTIES OF HANKEL IDEALS

Proposition 2.1. *Let G_1, G_2 be connected closed graphs on the vertex set $[m]$, respectively, $[n]$ and let G be the graph associated to the pair (G_1, G_2) . Then every maximal clique of G is of the form $[a + c - 1, b + d - 2]$ where $[a, b]$ is a maximal clique of G_1 and $[c, d]$ is a maximal clique of G_2 .*

Proof. Let $[p, q]$ be a maximal clique of G . Then $p = i + k - 1$, $q = j + l - 2$ for some $\{i, j\} \in E(G_1)$ and $\{k, l\} \in E(G_2)$. We claim that $[i, j]$ is a maximal clique of G_1 and $[k, l]$ is a maximal clique of G_2 . We need to prove only the first part of the claim since the second part can be proved in a similar way.

Since $\{i, j\} \in E(G_1)$ and G_1 is closed, it follows that $[i, j]$ is a clique of G_1 . Let us assume that $[i, j]$ is not a maximal clique. Then there exists $u \in V(G_1)$, $u < i$, such that $\{u, j\} \in E(G_1)$ or there exists $v \in V(G_1)$, $v > j$, such that $\{i, v\} \in E(G_1)$. In the first case, we get $\{u + k - 1, j + l - 2\} \in E(G)$, which is impossible since $u + k - 1 < i + k - 1$ and $[i + k - 1, j + l - 2]$ is a maximal clique of the closed graph G . Similarly, if $\{i, v\} \in E(G_1)$ for some $v > j$, we get $\{i + k - 1, v + l - 2\} \in E(G)$, again a contradiction by the same argument as above. \square

Remarks 2.2. (1) It is clear that if $[a, b]$ is a maximal clique of G_1 and $[c, d]$ is a maximal clique of G_2 , then $[a + c - 1, b + d - 2]$ is a clique of G . But it might happen that $[a + c - 1, b + d - 2]$ is not a maximal one. For example, let G_1, G_2 be closed graphs on the vertex set $[5]$ with the maximal cliques $F_{11} = [1, 3]$, $F_{12} = [2, 4]$, $F_{13} = [3, 5]$ and $F_{21} = [1, 3]$, $F_{22} = [2, 5]$, respectively. One can easily see that the maximal cliques $F_{13} = [3, 5]$ and $F_{21} = [1, 3]$ give the clique $[3, 6]$ in the associated graph G which is not maximal. Actually, the maximal cliques of G are $[1, 3]$, $[2, 6]$, $[3, 7]$, and $[4, 8]$.

(2) The cliques $[a, b]$ and $[c, d]$ in the above proposition are not necessarily uniquely determined by the maximal clique of G . For example, let G_1, G_2 be line graphs on the vertex set $[3]$. The associated graph G is again a line graph on the vertex set $[4]$. Then, the maximal clique $[2, 3]$ in G can be obtained either by "adding" the clique $[1, 2]$ of G_1 with $[2, 3]$ of G_2 or by using $[2, 3]$ from G_1 and $[1, 2]$ from G_2 .

The following theorem collects the main properties of the ideal I_{G_1, G_2} . In the statement we use the well-known notation $\text{Ass}(I)$ and $\text{Min}(I)$ for the associated prime ideals and, respectively, minimal prime ideals of I .

Theorem 2.3. *Let G_1, G_2 be connected closed graphs on the vertex sets $[m]$, respectively $[n]$. Then:*

(1) I_{G_1, G_2} is a prime ideal if and only if G_1 and G_2 are complete graphs.

(2) If at least one of the graphs G_1, G_2 is not complete, then

$$\text{Ass}(I_{G_1, G_2}) = \text{Min}(I_{G_1, G_2}) = \{I_{K_m, K_n}, (x_2, \dots, x_{m+n-2})\}.$$

(3) I_{G_1, G_2} is a set-theoretical complete intersection.

(4) I_{G_1, G_2} is a radical ideal if and only one of the following holds:

- (a) $G_1 = K_m$ and either $G_2 = K_n$ or $\Delta(G_2) = \langle [1, n-1], [2, n] \rangle$;
- (b) $G_2 = K_n$ and either $G_1 = K_m$ or $\Delta(G_1) = \langle [1, m-1], [2, m] \rangle$;

(5) The following statements are equivalent:

- (a) I_{G_1, G_2} has a linear resolution;
- (b) All powers of I_{G_1, G_2} have a linear resolution;
- (c) I_{G_1} and I_{G_2} have a linear resolution;
- (d) G_1 and G_2 are complete graphs.

Proof. By Theorem 1.1, we know that $I_{G_1, G_2} = I_G$ where G is the associated graph of the pair G_1, G_2 . Hence, in all the statements, we may replace I_{G_1, G_2} by I_G .

(1) If $G_1 = K_m$ and $G_2 = K_n$, then $G = K_{m+n-2}$, and the claim is known. Conversely, let I_G be a prime ideal. Then, by [1, Theorem 2.2], it follows that G is a complete graph. Hence G is the clique $[1, m+n-2]$. By Proposition 2.1, it follows that there exist $[a, b]$ maximal clique in G_1 and $[c, d]$ maximal clique in G_2 such that $[a+c-1, b+d-2] = [1, m+n-2]$. This equality implies that $G_1 = K_m$ and $G_2 = K_n$.

(2) follows by (1) and [1, Theorem 2.2].

(3) This is direct consequence of [1, Corollary 2.4].

(4) Let us assume that $G_2 = K_n$ and the facets of the clique complex of G_1 are $[1, m-1]$ and $[2, m]$. Then one easily sees that the facets of the clique complex of G are $[1, m+n-3]$ and $[2, m+n-2]$. Hence, by using [1, Proposition 2.3], it follows that I_G is a radical ideal. Let now I_G be a radical ideal which is not prime. By [1, Proposition 2.3] it follows that G has two maximal cliques, namely $[1, m+n-3]$ and $[2, m+n-2]$. Let $[a, b]$ and $[c, d]$ be maximal cliques in G_1 , respectively G_2 , such that $[a+c-1, b+d-2] = [1, m+n-3]$. This equality implies that $[a, b] = [1, m-1]$ and $[c, d] = [1, n]$ or $[a, b] = [1, m]$ and $[c, d] = [1, n-1]$. Hence, G_1 or G_2 is a complete graph. Let us choose, for instance, $G_2 = K_n$, and assume that $G_1 \neq K_m$. By the form of the cliques of G , it follows that G_1 has the maximal cliques $[1, m-1]$ and $[2, m]$.

The equivalence of the statements in (6) follows by applying [1, Proposition 2.6] and statement (1) in this theorem. \square

In [1, Theorem 2.7] it was shown that, for any closed graph H on the vertex set $[n]$, the regularity of $K[x_1, \dots, x_{n+1}]/I_H$ is at most the number of maximal cliques of H . Therefore, we get $\text{reg}(K[x_1, \dots, x_{n+1}]/I_H) \leq n-1$. If equality holds in this inequality, it follows that H must be the line graph on $[n]$. Conversely, if H is the line graph, then I_H is a complete intersection, hence the Koszul complex gives the minimal graded free resolution of $K[x_1, \dots, x_{n+1}]/I_H$. This implies that $\text{reg}(K[x_1, \dots, x_{n+1}]/I_H) = n-1$.

In our context we get the following result.

Proposition 2.4. *Let G_1, G_2 be connected closed graphs on the vertex set $[m]$, respectively, $[n]$. Then $\text{reg}(S/I_{G_1, G_2}) \leq m + n - 2$ and the equality holds if and only if G_1 and G_2 are line graphs.*

Proof. The inequality follows by Theorem 1.1. If G_1 and G_2 are line graphs, one may easily check that the associated graph G is a line graph too, hence $\text{reg}(S/I_G) = m + n - 2$. Let us now assume that $\text{reg}(S/I_G) = m + n - 2$. Thus, I_G is the line graph on $[m + n - 2]$, hence its maximal cliques are $[i, i + 1]$ for $1 \leq i \leq m + n - 3$. Let us assume, for example, that G_1 is not a line graph. Therefore, G_1 has at least one maximal clique $[a, b]$ with $b > a + 1$. Then, for any maximal clique $[c, d]$ of G_2 , $[a + c - 1, b + d - 2]$ is a clique of G . But $b + d - 2 > (a + c - 1) + 1$, hence G cannot be a line graph. \square

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